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Strong Conservative Form of the Incompressible Navier–Stokes Equations in a Rotating Frame with a Solution Procedure

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The Navier–Stokes equations in a rotating frame of reference have been formulated in the so-called strong conservative form, i.e., without the traditional source terms, viz., the Coriolis and centrifugal forces. These equations have been coupled with the continuity equation by using the modified artificial compressibility method in order to develop an implicit numerical scheme that has third order accuracy in space and second order accuracy in time. This scheme uses the Roe fluxes and the MUSCL extrapolation techniques to obtain the desired accuracies in space and the backward Euler formula to obtain the desired time accuracy. The flux Jacobians and their eigensystem are presented which are required in the development of the numerical scheme. The resulting scheme was used to solve the Ekman boundary layer problems with (a) no slip and (b) applied surface stress boundary conditions and excellent agreement between computed and exact solutions has been obtained, supporting the new formulation of the governing equations as well as the solution procedure.

1. INTRODUCTION

Fluid flow problems can be analyzed by formulating the governing equations in either an inertial frame or a non-inertial frame. The frame of reference is usually chosen depending upon the problem at hand. Thus one would normally choose an inertial frame to study the flow over a stationary airfoil in a uniform flow and a non-inertial frame to study the flow through a compressor or turbine as well as oceanographic and atmospheric flows. In the later case the particular non-inertial frame would be a rotating frame. In the general case of non-inertial frame it is possible to have coordinate lines that rotate, translate, and deform. While casting the governing equations to describe the flow in a non-inertial frame there are two choices regarding the velocity vector. Either it can be the velocity vector with respect to the absolute (inertial) frame, hereafter called the absolute velocity vector for brevity, or it can be the velocity vector with respect to the relative (non-inertial) frame, hereafter called the relative velocity vector for brevity. Depending upon this choice various formulations result.

The classical formulation of a flow in a rotating frame (see for example, Gill [1]) utilizes the relative velocity vector and casts the governing equations with respect to the rotating frame, resulting in two source terms, viz., the Coriolis force term and the centrifugal force term. It is possible to express the centrifugal force term as the gradient of a potential and so the Coriolis force remains as the only true source term in this formulation. For the case of a self-rotating gravitational body such as the earth, the centrifugal force is accounted for in its gravity. Until now this is the formulation used by the majority of researchers to solve oceanographic flow problems. However, as has been shown in this work a simple tensor identity can be used to cast the governing equations in a rotating frame in the so-called strong conservative form. This opens up new possibilities of building alternate numerical approaches to solving the governing equations in a rotating frame.

The classical formulation of the governing equations in a rotating frame is just one particular way of solving oceanographic flow problems. A time marching approach using this set of equations would solve for the components of the relative velocity vector at every time step. An alternate approach would be to solve the governing equations cast using the absolute velocity vector in the local time derivative term. A time marching scheme using this set of equations would solve for the components of the absolute velocity vector at every time step. The formulation presented by Vinokur [2] (see Eq. (2.9) below) utilizes the absolute velocity vector and casts the equations in a strong conservative form (i.e. without source terms) to describe the flow in a general non-inertial frame. This formulation uses the unsteady Eulerian coordinates for describing a compressible flow from which the analogous equations for incompressible flows can be derived by setting the density to be constant.

The classical formulation in the rotating frame as well as the formulation of Vinokur [2] expresses the governing equations using the Eulerian description, and that provides an additional choice in the formulation through the local time derivative. The formulation of Vinokur [2] expresses the local time derivative in the absolute frame, whereas the classical formulation of a flow in a rotating

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frame expresses the local time derivative in the rotating frame. Agarwal and Deese [3] have applied the formulation of Vinokur [2] for a rotating frame and expressed the local time derivative in the relative frame and put forth a formulation where only half of the Coriolis force is expressed as the source term and the other half is absorbed in the divergence term. The present work can be considered as an extension of the formulation of Agarwal and Deese [3] in the sense that the remaining half of the Coriolis force has also been absorbed into the divergence term so that the momentum equation of an incompressible flow in a rotating frame is cast in the strong conservative form. From a theoretical point of view all these formulations are equivalent. But from a numerical point of view there are differences between these formulations. For example, a steady flow in a rotating frame would appear as an unsteady flow in an absolute frame. Thus time accurate solutions are needed if one is expressing the local time derivative with respect to the absolute frame with absolute velocity components. There exists a large amount of literature (see, for example, Yee [4]) that discusses the effect of source terms on a numerical scheme. Experience reported in Ref. [4] by various authors has shown that it is always advantageous to avoid the source terms, if possible, and cast the governing equations in the strong conservative form.

From a numerical point of view, one of the advantages of expressing the Coriolis term in a conservative form, in other words, in a divergence form, is that it fits in a natural manner in a finite volume scheme, since fluxes are evaluated at the cell faces in such a scheme whereas source terms need to be evaluated at cell centers. A more important advantage is that in a higher order numerical approximation of the fluxes the Coriolis term naturally enters the flux Jacobian matrix, as can be seen from the Appendix, and a higher order representation for the Coriolis term is thus possible.

It is shown that the momentum equations in the rotating frame can be cast either with the absolute velocity vector appearing in the local time derivative or with the relative velocity vector appearing in the local time derivative, where, in both cases, the local time derivative itself is expressed with respect to the rotating frame. Both of the formulations are numerically examined together with the formulation where half of the Coriolis force is treated as a source term. The resulting momentum equations are coupled with the continuity equation using the modified artificial compressibility method.

2. MATHEMATICAL FORMULATION

The momentum equations governing the (oceanic) flows over earth, which is a self-rotating gravitational body, in non-dimensional tensor invariant form is given by (see, for example, Gill [1])

$$\frac{\hat{\partial}\mathbf{v}}{\partial\tau} + \nabla\cdot\left[\mathbf{vv} + p\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] + 2\mathbf{\Omega}\times\mathbf{v} + \mathbf{b} = 0, \tag{2.1}$$

where $\mathbf{v} = \mathbf{v}^*/U_0$ is the non-dimensional velocity vector with respect to the rotating frame, $\tau = tU_0/L$ is the non-dimensional time, $p = (p^* - p_0)/\varrho_0 U_0^2$ is the non-dimensional pressure, $\mathbf{\Omega}$ is the angular velocity of the rotating frame, $\tilde{\sigma}$ is the Stokes tensor, and $\mathbf{b}$ is the body force. Re_0 is the Reynolds number, and $\mathrm{Re}_0 = \varrho_0 U_0 L/\mu_0$, where ϱ_0 is a reference density, U_0 is a reference velocity, L is a reference length, and μ_0 is a reference coefficient of viscosity. A tilde over a quantity denotes that it is a tensor and boldface denotes that it is a vector. The Stokes tensor is given by

$$\tilde{\sigma} = \mu(\nabla\mathbf{v} + \nabla^{\mathrm{T}}\mathbf{v}) \tag{2.2}$$

where $\mu = \mu^*/\mu_0$ is the non-dimensional coefficient of viscosity. The superscript "T" in Eq. (2.2) denotes the transpose operation. The only body force considered is that due to gravity and is given by $\mathbf{b} = \mathbf{n}/\mathrm{Fr}^2$, where Fr is the Froude number given by $\mathrm{Fr} = U_0/\sqrt{aL}$, where a is the acceleration due to gravity and $\mathbf{n}$ is the local normal to the earth's surface. In Eq. (2.1), $\hat{\partial}/\partial\tau$ denotes the local time derivative with respect to the rotating frame. In other words, if $\mathbf{i}_m$, $m = 1, 2, 3$, are the Cartesian base vectors in the rotating frame then, by definition,

$$\frac{\hat{\partial}\mathbf{v}}{\partial\tau} = \frac{\hat{\partial}(v_m\mathbf{i}_m)}{\partial\tau} = \frac{\hat{\partial}v_m}{\partial\tau}\mathbf{i}_m \tag{2.3}$$

It can be easily verified that

$$\Omega\times\mathbf{v} = \mathbf{v}\cdot\nabla(\Omega\times\mathbf{r}), \tag{2.4}$$

where $\mathbf{r}$ is the radius vector from the origin of the rotating frame. Using the tensor identity for any two vectors $\mathbf{a}$ and $\mathbf{b}$ (see Morse and Feshbach [5]),

$$\nabla\cdot(\mathbf{ab}) = \mathbf{a}\cdot(\nabla\mathbf{b}) + \mathbf{b}(\nabla\cdot\mathbf{a}), \tag{2.5}$$

and the fact that $\nabla\cdot\mathbf{v} = 0$, which follows from the conservation of mass, it is seen that

$$\mathbf{\Omega}\times\mathbf{v} = -\nabla\cdot(\mathbf{vw}), \tag{2.6}$$

where $\mathbf{w} = -\mathbf{\Omega} \times \mathbf{r}$. Therefore, Eq. (2.1) can be written as

$$\frac{\hat{\partial}\mathbf{v}}{\partial\tau} + \nabla \cdot \left[\mathbf{v}(\mathbf{v} - 2\mathbf{w}) + p'\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] = 0 \tag{2.7}$$

Note that the body force has been combined with the pressure by using the body force potential in the manner prescribed by Beddhu *et al.* [6], that is, $p' = p + \chi/\mathrm{Fr}^2$, where χ is the body force potential due to gravity. Equation (2.7) is the strong conservative formulation of the Navier–Stokes equation for incompressible flows in a rotating frame. To the authors' best knowledge this is the first time the Navier–Stokes equations have been presented in a rotating frame without both source terms. Note that such a formulation is not possible for compressible flows since $\nabla \cdot \mathbf{v} \neq 0$. The continuity equation in the modified artificial compressibility method [6] is given by

$$\frac{\hat{\partial}p'}{\partial\tau} + \beta\nabla \cdot \mathbf{v} = 0 \tag{2.8}$$

where β is the artificial compressibility parameter.

Even though Eqs. (2.8) and (2.7) form a complete set of governing equations for solving the oceanic flow problems, further insight into the alternative formulations of the momentum equation available for solving the geophysical flow problems can be gained by looking at an alternate derivation, starting from the governing equations with respect to an arbitrary non-inertial frame. Since a generalized setting is considered, the strong conservative form of the governing equations is presented for the case of a rotating frame in a gravitational field (a turbomachinery problem, for example), in addition to the case of a self-gravitating, rotating body such as the earth.

The momentum equation for viscous, incompressible flows in a non-inertial frame of reference in a gravitational field, in a non-dimensional, vector invariant form is given by (see, for example, Warsi [7])

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\mathbf{u})}{\partial\tau} + \nabla \cdot \left[\mathbf{v}\mathbf{u} + p\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] + \mathbf{b} = 0, \tag{2.9}$$

where $\sqrt{g}$ is the Jacobian of the coordinate transformation, $\mathbf{u} = \mathbf{u}^*/U_0$ is the non-dimensional velocity vector in the absolute frame, $\mathbf{v} = \mathbf{u} + \mathbf{w}$ is the non-dimensional velocity vector relative to the moving frame, $\mathbf{w}$ is the non-dimensional grid speed vector, and other quantities are as defined previously. The Stokes tensor is given by

$$\tilde{\sigma} = \mu(\nabla\mathbf{u} + \nabla^T\mathbf{u}). \tag{2.10}$$

It must be noted here that Warsi [7] follows the linear transformation representation (see Truesdell and Noll [8]) for representing tensors whereas this work has adopted the dyadic product representation (see Morse and Feshbach [5]) for representing tensors. Hence, the equations found in Ref. [7] are suitably modified to fit the representation adopted in this work.

Since a rotating frame is a particular case of non-inertial frames for which Eq. (2.9) is applicable, it must be possible to derive the momentum equation in a rotating frame from Eq. (2.9). However, the concept of grid speed is not valid with respect to an observer situated in the rotating frame since the grid does not move with respect to him/her. Following Warsi [7], instead of considering $\mathbf{w}$ as the grid speed, one poses the question of what form of $\mathbf{w}$ in Eq. (2.9) would result in the Navier–Stokes equations in a rotating frame. It is an exercise problem in Ref. [7] to show that substituting $\mathbf{w} = -\mathbf{\Omega} \times \mathbf{r}$, where $\mathbf{\Omega}$ is the angular velocity of the rotating frame and $\mathbf{r}$ is the radius vector from the origin of the rotating frame in Eq. (2.9), results in the classical rotating frame equation in a gravitational field, i.e., the centrifugal force term, $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, which has to be added to the left-hand side of Eq. (2.1). Hence, in order to arrive at Eq. (2.1) this term has to be subtracted from Eq. (2.9) to obtain

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\mathbf{u})}{\partial\tau} + \nabla \cdot \left[\mathbf{v}\mathbf{u} + p\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] + \mathbf{b} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = 0. \tag{2.11}$$

It is emphasized that now Eq. (2.11) is applicable only to self-gravitating, rotating bodies like the earth. Note that the time derivative in Eq. (2.11) is still with respect to the inertial frame. Equation (2.11) is the appropriate equation to be used when the rotation of the earth is prescribed through grid motion. In addition to the rotation of the earth, one can also include the effects of the evolving free surface using moving grids in a natural manner. For the case of a rotating frame with a constant angular velocity, $\mathbf{\Omega}$, using the relations given in Section 3.10B of Ref. [7], it can be proved easily that

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\mathbf{u})}{\partial\tau} = \frac{\hat{\partial}\mathbf{u}}{\partial\tau} + \mathbf{\Omega} \times \mathbf{u} \tag{2.12}$$

where $\hat{\partial}/\partial\tau$ denotes the local time derivative with respect to the rotating frame. Equation (2.11) can now be re-written, using Eq. (2.12), as

$$\frac{\hat{\partial}\mathbf{u}}{\partial\tau} + \nabla\cdot\left[\mathbf{vu} + p'\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] + \mathbf{\Omega}\times\mathbf{v} = 0, \quad (2.13)$$

where $\mathbf{v} = \mathbf{u} + \mathbf{w} = \mathbf{u} - \mathbf{\Omega}\times\mathbf{r}$ is the velocity with respect to the rotating frame. Note that the body force has been combined with the pressure by using the body force potential as before. Substitution of Eq. (2.6) in Eq. (2.13) results in

$$\frac{\hat{\partial}\mathbf{u}}{\partial\tau} + \nabla\cdot\left[\mathbf{v}(\mathbf{u}-\mathbf{w}) + p'\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] = 0. \quad (2.14)$$

Equation (2.14) is an alternate strong conservative form of the Navier–Stokes equations in a rotating frame, applicable for a self-gravitating, rotating body like the earth. Note that Eq. (2.7) can be recovered from Eq. (2.14) by substituting $\mathbf{u} = \mathbf{v} - \mathbf{w} = \mathbf{u} + \mathbf{\Omega}\times\mathbf{r}$. The main difference between Eqs. (2.7) and (2.14) is that in a time marching approach one would solve for the relative velocity components using Eq. (2.7), whereas one would solve for the absolute velocity components using Eq. (2.14). The continuity equation in the modified artificial compressibility method [6] is given by

$$\frac{\hat{\partial}p'}{\partial\tau} + \beta\,\nabla\cdot\mathbf{u} = 0, \quad (2.15)$$

where β is the artificial compressibility parameter.

So far, the discussion is focused on deriving the governing equations appropriate for geophysical flows. In other words, the governing equations (2.7) and (2.14) are appropriate for flows over bodies, such as the earth, which are self-rotating and self-gravitating bodies. That is, they rotate on their own accord and have their own gravity fields. However, there are bodies of practical interest, such as a turbomachine, that rotate in an external gravitational field. In the case of the earth, the centrifugal force that arises due to the rotation of the earth is combined with Newton's law of gravitation to arrive at an effective value of the acceleration due to gravity, viz, 9.81 m/s^2. In the case of a turbomachine in an external gravitational field, however, one has to account for the centrifugal force created by the rotating parts of the machine explicitly, regardless of whether one considers the effect of the external gravitational field or not. In the next paragraph the governing equations are given for the case of a rotating frame in a gravitational field. To fix ideas, one can consider the rotating frame to be a turbomachine. If one is not interested in accounting for gravity then all one needs to do is to replace p' by p in Eq. (2.16).

Agarwal and Deese [3] derived the compressible Euler equations analogous to Eq. (2.13) for a rotating frame in a gravitational field (whose effects were neglected) and so had $\mathbf{\Omega}\times\mathbf{u}$ instead of $\mathbf{\Omega}\times\mathbf{v}$ as the source term. For the sake of completeness it is mentioned here that for the general case of a rotating frame in a gravitational field the analogous strong conservative form of the momentum equation is

$$\frac{\hat{\partial}\mathbf{u}}{\partial\tau} + \nabla\cdot\left[\mathbf{vu} - \mathbf{uw} + p'\tilde{I} - \frac{1}{\mathrm{Re}_0}\tilde{\sigma}\right] = 0. \quad (2.16)$$

The starting point for obtaining Eq. (2.16) is Eq. (2.9). Note that the second term within the divergence operator in Eq. (2.16) is $-\mathbf{uw}$ whereas it is $-\mathbf{vw}$ in Eq. (2.14). Note also that $-\mathbf{uw} = -\mathbf{vw} + \mathbf{ww}$, and it can be easily verified using the relations provided earlier in this paper that $\nabla\cdot[\mathbf{ww}] = \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$. Since the case being considered in this paragraph is a rotating frame in a gravitational field (like a turbomachine, say), as opposed to the case of a self-rotating gravitational field (like the earth) which was considered earlier, the term $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$ which was subtracted earlier in Eq. (2.11) should be added back to it, and that is what leads one to Eq. (2.16).

The fully conservative formulation of the momentum equation is given in the compact vector and tensor notations, thus far. However, in order to solve the equations, numerically or otherwise, one has to write the momentum equation in its component form. When resolving the momentum equation into component form one is presented with many choices. These choices arise due to the fact that the vector and tensor quantities can be expressed with respect to any set of coordinates independent of the coordinates one chooses to express the divergence operator itself. Traditionally, however, the set of coordinates chosen for resolving the vector and tensor quantities is the same as the one chosen for expressing the divergence operator. Thus, Cartesian velocity components are chosen when the divergence operator is expressed in Cartesian coordinates, cylindrical components are chosen when the divergence operator is expressed with respect to cylindrical coordinates, and so on. The problem, for example, with choosing cylindrical components of the vector and tensor quantities when expressing the divergence operator in cylindrical coordinates is that Christoffel symbols appear explicitly, thereby preventing the conservative formulation in the component form. Therefore, if one wants to come up with a conservative formulation in the component form also there is only one choice.

That choice is to express the vector and tensor quantities in Cartesian components, no matter what coordinates are chosen to express the divergence operator. Since non-orthogonal curvilinear coordinates are the most general coordinates, the divergence operator is expressed with respect to such a coordinate system in this paper. The resulting equations are given in (A.1) (see the Appendix). The code UNCLE.OMAS (*un*steady *c*omputation of fie*l*d *e*quations–*o*ld *m*an *a*nd the *s*ea) is written to solve Eqs. (A.1). Thus, suppose one is interested in the flow over a sphere. Then one can construct a grid based on spherical coordinates and the appropriate metrics will automatically be computed. However, the real advantage of this approach is that one does not have to create a grid based on spherical coordinates. As long as the body shape is maintained spherical, any set of coordinate lines can be created, analytically or numerically, and the same code can be used to solve the flow field.

Even though Eq. (2.9) (after expressing the body force in terms of the body force potential) and Eq. (2.16) are in fully conservative form and a time marching scheme in both cases would solve for the absolute velocity components, the important difference between them is the position of the observer. While in the case of Eq. (2.9) the oberver is situated in the inertial frame, he/she is situated in the rotating frame in the case of Eq. (2.16). Thus the grid remains stationary in the case of Eq. (2.16) whereas the grid has to be moved and all the metrics need to be recomputed at each time step in the case of Eq. (2.9). Steady flows in the rotating frame can be computed using time inaccurate schemes using Eq. (2.16), whereas they require time accurate computation of Eq. (2.9).

A time marching upwind scheme for the set of equations (2.14) and (2.15) would typically solve for the pressure and the Cartesian components of the absolute velocity vector. Either one can solve the set of equations (2.14) and (2.15), or the set (2.7) and (2.8), by the numerical method presented in the following section. For both sets remarkably similar sets of eigensystems are derived. These eigensystems again differ from that derived by Taylor [9] for Eq. (2.9) only slightly which results in minimum code modifications.

The solution procedure for the set of equations (2.14) and (2.15) is called the absolute-velocity procedure and that for the set of equations (2.7) and (2.8) is called the relative-velocity procedure. Because of the choice of the equations the solution procedure as presented here is valid for geophysical flows only. For turbomachinery type flows one could derive analogous procedures using Eq. (2.16) instead of Eq. (2.15). An important element in the present formulation is the construction of the inviscid fluxes at the cell interfaces. The theory behind the construction of the inviscid fluxes has been well established by Roe [10], van Leer [11], and others. Therefore, only the tools needed for constructing the inviscid fluxes are provided in Appendix A. The numerical method to be described has been presented in detail elsewhere [9, 12–14]. Only a brief description is given in the next section.

3. NUMERICAL PROCEDURE

The numerical scheme used in this study is similar to that proposed by Pan and Chakravarthy [12] and is discussed in detail by Taylor [9] and Whitfield and Taylor [13]. An extensive discussion of the methodology applicable to two-dimensional flows has been presented by Whitfield and Taylor [14]. The approach taken in this work is to solve Eq. (A.1) implicitly using the discretized Newton-relaxation (DNR) scheme [13], where the fluxes at the cell faces are obtained using the Roe scheme [10] with higher order accuracy achieved using the MUSCL approach (van Leer [11]; Whitfield and Taylor [14]). Writing Eq. (A.1) in the discrete form, one has

$$\begin{aligned}
&\frac{3\,Q^{n+1} - 4\,Q^{n} + Q^{n-1}}{2\,\Delta\tau} \\
&\quad + F^{n+1}_{i+1/2} - F^{n+1}_{i-1/2} + G^{n+1}_{j+1/2} - G^{n+1}_{j-1/2} \\
&\quad + H^{n+1}_{k+1/2} - H^{n+1}_{k-1/2} + F^{v^{n+1}}_{i+1/2} - F^{v^{n+1}}_{i-1/2} \\
&\quad + G^{v^{n+1}}_{j+1/2} - G^{v^{n+1}}_{j-1/2} + H^{v^{n+1}}_{k+1/2} - H^{v^{n+1}}_{k-1/2} = 0
\end{aligned} \tag{3.1}$$

where $F^{n+1}_{i+1/2} = F(Q^{n+1}_{i-1}, Q^{n+1}_{i}, Q^{n+1}_{i+1}, Q^{n+1}_{i+2})$ and so on. Note that for a higher order flux representation $F^{n+1}_{i+1/2}$ depends on Q^{n+1}_{i-1} and Q^{n+1}_{i+2} as well. If Eq. (3.1) is expanded for each grid cell, a system of algebraic equations are obtained in terms of q^{n+1} at each grid cell where $q^{n+1} = Q^{n+1}/\sqrt{g}^{n+1}$. Strictly speaking, F^{n+1} is a function of both q^{n+1} and the metrics at $n+1$. Since the metrics at $n+1$ are known, no linearization needs to be done with respect to the metrics. Hence Eq. (3.1) is regarded as a function of q^{n+1} alone. In functional form, Eq. (3.1) can be represented as

$$X(q^{n+1}) = 0. \tag{3.2}$$

Solving Eq. (3.2) involves finding the roots of a system of non-linear algebraic equations. Using Newton's method, the solutions of Eq. (3.2) are obtained from the linear equations

$$\left(\frac{\partial X}{\partial q}\right)^{n+1,m}(q^{n+1,m+1}-q^{n+1,m}) = -X(q^{n+1,m}). \quad (3.3)$$

In order to limit the band width of the matrix, the operator $(\partial X/\partial q)$ is obtained using higher order fluxes in a special manner and is rearranged along the lines of Whitfield and Taylor [14] into a strong diagonal form. The viscous flux Jacobians are obtained using the thin layer approximation, whereas the residue $X(q^{n+1,m})$ contains all the viscous terms. Within each Newton iteration, symmetric Gauss–Seidel passes are used. The resulting algorithm is termed the discretized Newton-relaxation procedure. When the iteration in m converges, q^{n+1} is obtained and the calculation procedure is extended to the next time level. As the iteration in m converges, the LHS of Eq. (3.3) goes to zero. Hence time accuracy is introduced into the scheme by multiplying the local time derivative term in $X(q^{n+1,m})$ by a conditioning matrix I_a where $I_a = \text{diag}(0, 1, 1, 1)$. The inviscid fluxes on the RHS of Eq. (3.3) are obtained by using a third order MUSCL-type flux and the viscous fluxes by using a second order central differencing.

4. RESULTS

Ekman boundary layer solutions are classical solutions of the equations that govern geophysical flows. A detailed description of these problems and their solutions can be found in Pedlosky [15]. A Cartesian coordinate system xyz is introduced in a rotating frame such that the y-axis coincides with the axis of rotation. A geostrophic flow is assumed to be in the z-direction. In the far field, the flow is geostrophic with the following values for the non-dimensional quantities: $u = 0$, $v = 0$, $w = 1$, and $\partial p/\partial x = -2$. Here the velocity components are with respect to the rotating frame. The velocity components in the absolute frame are obtained by adding the corresponding components of $\mathbf{\Omega} \times \mathbf{r}$ to the components of the relative velocity vectors. Suppose NI is the maximum number of points in the x-direction (I-direction). In the cell-centered finite volume formulation, Δq's, where $\Delta q = q^{n+1} - q^n$, with q any flow variable, are updated from cell 2 to cell NI. Boundary conditions are specified in the fictitious (or phantom) cells at 1 and $NI + 1$. When solving for Δq's in cells 2 to NI using the upwind scheme, the Δq's at cells 1 and $NI + 1$ are required (Fig. 1). Usually these quantities are taken as zero. In order to impose the one-dimensionality of the problem and to eliminate any bias introduced by the upwind scheme, the following periodicity conditions were used: $X_1 = X_{NI}$ and $X_{NI+1} = X_2$. A similar treatment is done in the z-direction (K-direction) also. The Reynolds number was taken to be 10,000. Since Ekman boundary layers correspond to the rigid lid condition the Froude number has no effect on the solution.

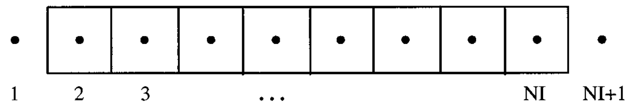

FIG. 1. Schematic showing the phantom cells 1 and $NI + 1$.

Three procedures were used to compute the velocity profiles: (1) the absolute-velocity procedure, (2) the relative-velocity procedure, and (3) the source-term procedure where the source term in Eq. (2.13) is added to the code developed by Taylor [9]. The absolute-velocity procedure and the relative-velocity procedure converge in the same manner in 2000 cycles to the exact solution using a time step of 0.05. At viscous boundaries the pressure is extrapolated from the interior. The source-term procedure with a time step of 0.005 diverged after 10,000 cycles when the pressure was extrapolated from the interior. However, when the pressure was prescribed as $-2x$ at the viscous boundaries it converged to the correct solution. This indicates that the source term seems to corrupt the pressure gradient at the wall. The results shown were obtained using the relative-velocity procedure.

4.1. *Viscous Wall*

The computational domain is a rectangular parallelepiped bounded by the planes $x = 0$, $x = 1$, $z = 0$, $z = 1$, $y = 0$, and $y = 5$. Uniform spacing was used in the x and z directions and the grid lines were stretched in the y-direction with a spacing of 0.001 near the viscous surface ($y = 0.0$), see Fig. 2. The grid size is 51, 101, 5 points in the x, y, and z directions respectively. Linear extrapolation boundary conditions were used on all the side boundaries. This treatment of the side wall boundaries was chosen since it is known that the exact solution for the velocity components is independent of the x and z directions and

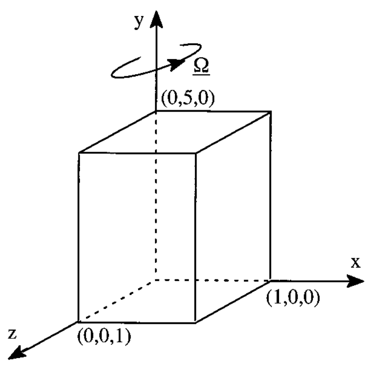

FIG. 2. Physical domain (not to scale).

the pressure is a linear function of x. The inviscid (i.e., geostrophic) solution was imposed at the top boundary. At the bottom boundary, no-slip velocity and zero pressure gradient boundary conditions were imposed. Starting from an initial condition of geostrophic flow everywhere, convergence to the exact solution was achieved in 2000 cycles using a time step of 0.05. Figure 3 shows the comparison of the computed results with the exact solution. Note that the stretched y-coordinate, $\bar{y} = y/\sqrt{\text{Re}}$, is used in the figure. It can be seen that the agreement is excellent.

4.2. *Free Surface with an Applied Shear Stress*

The computational domain and the number of points are the same as before except that the $y = 5.0$ boundary was used to impose the wind stress and consequently the grid points were packed near that boundary in the y-direction. The lateral boundary conditions are the same. At the bottom boundary the known inviscid (i.e., geostrophic) flow was imposed. The main idea in the application of the viscous free surface stress condition is to use a set of local orthonormal coordinates at every grid point on the free surface to implement the stress boundary condition. This is because the stress boundary conditions take the simplest form in an orthonormal coordinate system. Figure 4 shows the comparison of the computed results with the exact solution. To conform with the figure given in Ref. [15], the ordinate in Fig. 4 was obtained as $\bar{y} = (5.0 - y)/\sqrt{\text{Re}}$ during the postprocessing of the results. It can be seen that the agreement is excellent.

5. CONCLUSION

A new formulation of the Navier–Stokes equations in the rotating frame using relative velocity components has been presented in which no source terms appear. This leads to a numerical scheme that is very similar to the one formulated for the Navier–Stokes equations in the absolute frame using absolute velocity components. For geophysical flows both the absolute velocity procedure and the relative velocity procedure seem to be equally efficient. The relative velocity procedure is being applied to the computation of the flow field in the Atlantic ocean with the wind stresses prescribed from the European Center for Medium Range Weather Forecasts dataset [16]. Application of the Navier–Stokes equations to the computation of oceanic flows is still in an infantile stage. For a related work that uses a non-hydrostatic model to study the evolution of plumes one may refer to Helen and Marshall [17].

APPENDIX A: FLUX FORMULATION

The absolute velocity procedure is dealt with in detail. The relative velocity procedure can be similarly treated, and it is briefly outlined following the discussion of the absolute velocity procedure.

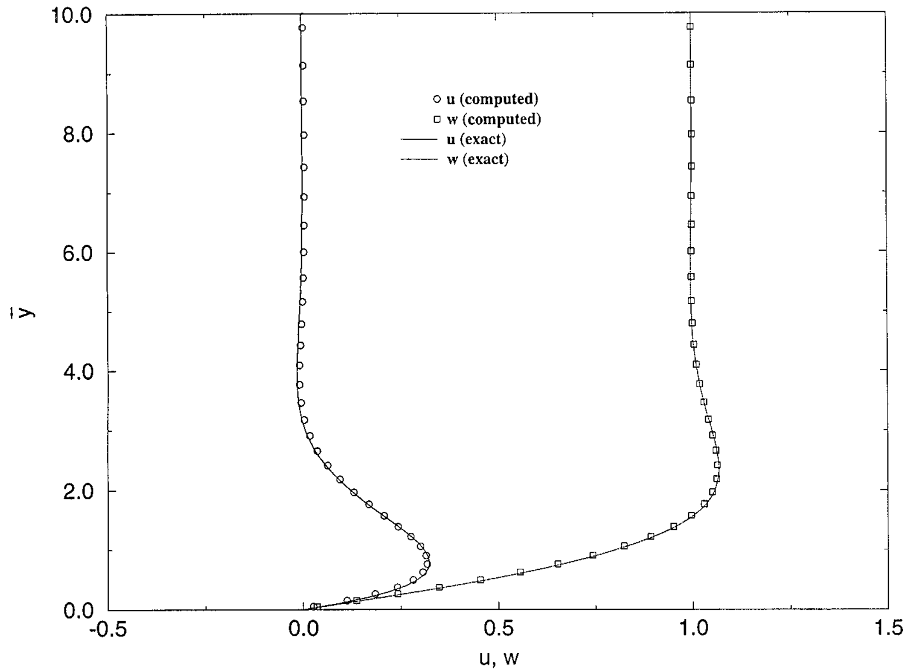

FIG. 3. Ekman boundary layer with a no slip wall: Re = 10000; rotation is about the y-axis.

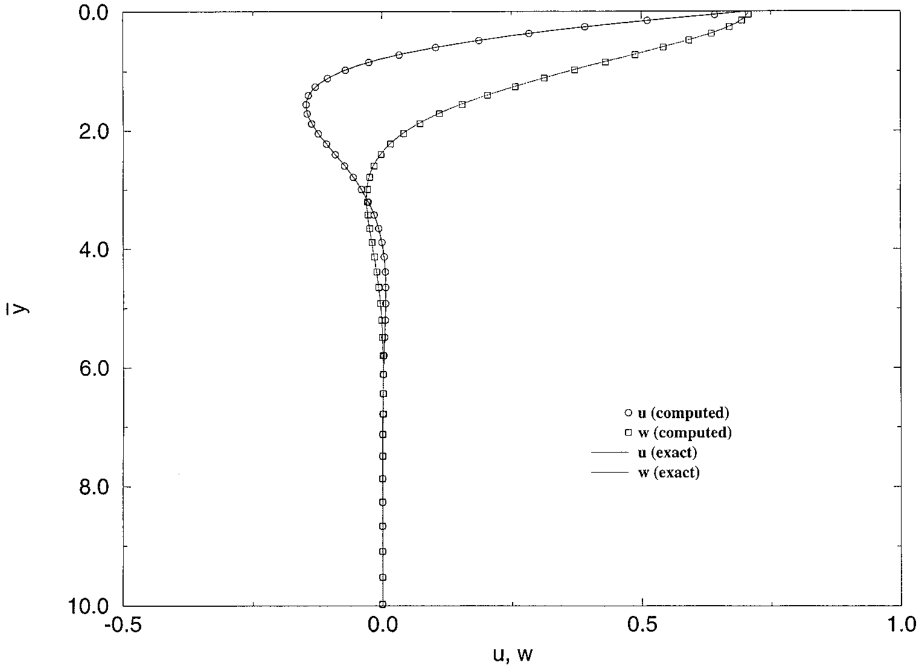

FIG. 4. Ekman boundary layer with applied shear stress at the free surface: Re = 10000; rotation is about the y-axis.

A.1. *Absolute-Velocity Procedure*

Equations (2.14) and (2.15) can now be expressed in a curvilinear coordinate system (ξ, η, ζ, τ), using the so-called partial transformation where all the tensor and vector quantities within the divergence terms are expressed with respect to the underlying Cartesian coordinates whereas the divergence itself is expressed in curvilinear coordinates, which can further be cast into the numerical vector form which results in

$$\frac{\partial Q}{\partial \tau} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} + \frac{\partial H}{\partial \zeta} + \frac{\partial F^v}{\partial \xi} + \frac{\partial G^v}{\partial \eta} + \frac{\partial H^v}{\partial \zeta} = 0, \qquad \text{(A.1)}$$

where

$$Q = \sqrt{g}\begin{bmatrix} p' \\ u \\ v \\ w \end{bmatrix}; \qquad F = \sqrt{g}\begin{bmatrix} \beta u^1 \\ u'(u^1 + \xi_t) + p'\xi_x \\ v'(u^1 + \xi_t) + p'\xi_y \\ w'(u^1 + \xi_t) + p'\xi_z \end{bmatrix};$$

$$F^v = \sqrt{g}\begin{bmatrix} 0 \\ \sigma_{xx}\xi_x + \sigma_{xy}\xi_y + \sigma_{xz}\xi_z \\ \sigma_{xy}\xi_x + \sigma_{yy}\xi_y + \sigma_{yz}\xi_z \\ \sigma_{xz}\xi_x + \sigma_{yz}\xi_y + \sigma_{zz}\xi_z \end{bmatrix};$$

$$u^1 = u\xi_x + v\xi_y + w\xi_z;$$

u, v, and w are the components of the absolute velocity vector with respect to a Cartesian coordinate system; and u', v', and w' are the Cartesian components of the vector $\mathbf{u} - \mathbf{w}$; σ_{xx}, etc., are the Cartesian components of the Stokes tensor; and ξ_x, ξ_y, and ξ_z are the Cartesian components of the contravariant base vector grad ξ. Expressions for G and H are similar to F and can be obtained from F by replacing ξ with η and ζ respectively. Similarly, G^v and H^v can be obtained from F^v. By defining the flux Jacobians as $A = \partial F/\partial q$; $B = \partial G/\partial g$; $C = \partial H/\partial q$ and denoting the generic flux Jacobian by K, one obtains

$$K = \sqrt{g}\begin{bmatrix} 0 & \beta k_x & \beta k_y & \beta k_z \\ k_x & \theta_k + u'k_x & u'k_y & u'k_z \\ k_y & v'k_x & \theta_k + v'k_y & v'k_z \\ k_z & w'k_x & w'k_y & \theta_k + w'k_z \end{bmatrix} \qquad \text{(A.2)}$$

where $\theta_k = k_t + uk_x + vk_y + wk_z$ and $k_t = \mathbf{w} \cdot \mathbf{a}^k$ where $\mathbf{a}^k$ is the contravariant base vector on the k = constant face. When $k = \xi$, $K = A$; when $k = \eta$, $K = B$; and when $k = \zeta$, $K = C$. In order to find the eigenvalues of K, the following matrix M and its inverse M^{-1} are used to form the matrix $\kappa = MKM^{-1}$.

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{u'}{\beta} & 1 & 0 & 0 \\ \dfrac{v'}{\beta} & 0 & 1 & 0 \\ \dfrac{w'}{\beta} & 0 & 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{u'}{\beta} & 1 & 0 & 0 \\ -\dfrac{v'}{\beta} & 0 & 1 & 0 \\ -\dfrac{w'}{\beta} & 0 & 0 & 1 \end{bmatrix}$$

The matrix κ is given by

$$\kappa = \sqrt{g}\begin{bmatrix} \theta_k - 2k_t & \beta k_x & \beta k_y & \beta k_z \\ k_x + \dfrac{u'\theta_k}{\beta} & \theta_k & 0 & 0 \\ k_y + \dfrac{v'\theta_k}{\beta} & 0 & \theta_k & 0 \\ k_z + \dfrac{w'\theta_k}{\beta} & 0 & 0 & \theta_k \end{bmatrix}. \tag{A.3}$$

The eigenvalues of K and κ are the same since they are similar matrices. However, it is much easier to find the eigenvalues of κ rather than those of K, and these are found to be

$$\left.\begin{aligned} \lambda_{1,2} &= \theta_k \\ \lambda_3 &= \theta_k - k_t + c \\ \lambda_4 &= \theta_k - k_t - c \end{aligned}\right\}, \tag{A.4}$$

where

$$c = \sqrt{(\theta_k - k_t)^2 + \beta(k_x^2 + k_y^2 + k_z^2)}.$$

Following Taylor [9], in order to obtain the left and right eigenvectors of K, first the left and right eigenvectors of κ are obtained. They are the columns and rows of the following matrices respectively:

$$P_k = \begin{bmatrix} 0 & 0 & -\tilde{c}^{-} & -\tilde{c}^{+} \\ \tilde{x}1 & \tilde{x}2 & \phi 1 & \phi 1 \\ \tilde{y}1 & \tilde{y}2 & \phi 2 & \phi 2 \\ \tilde{z}1 & \tilde{z}2 & \phi 3 & \phi 3 \end{bmatrix}$$

$$P_k^{-1} = \frac{1}{\phi}\begin{bmatrix} 0 & 2\tilde{c}\phi 4 & -2\tilde{c}\phi 5 & 2\tilde{c}\phi 6 \\ 0 & -2\tilde{c}\phi 7 & 2\tilde{c}\phi 8 & -2\tilde{c}\phi 9 \\ \dfrac{\phi}{2\tilde{c}} & 2k_x\tilde{c}^{+} & 2k\tilde{c}^{+} & 2k_z\tilde{c}^{+} \\ -\dfrac{\phi}{2\tilde{c}} & -2k_x\tilde{c}^{-} & -2k_y\tilde{c}^{-} & -2k_z\tilde{c}^{-} \end{bmatrix},$$

where

$$\phi 1 = \tilde{k}_x + \frac{u'\tilde{\theta}_k}{\beta} \qquad \phi 2 = \tilde{k}_y + \frac{v'\tilde{\theta}_k}{\beta} \qquad \phi 3 = \tilde{k}_z + \frac{w'\tilde{\theta}_k}{\beta}$$

$$\phi 4 = \tilde{y}2\ \ \phi 3 - \tilde{z}2\ \ \phi 2 \qquad \phi 5 = \tilde{x}2\ \ \phi 3 - \tilde{z}2\ \ \phi 1 \qquad \phi 6 = \tilde{x}2\ \ \phi 2 - \tilde{y}2\ \ \phi 1$$

$$\phi 7 = \tilde{y}1\ \ \phi 3 - \tilde{z}1\ \ \phi 2 \qquad \phi 8 = \tilde{x}1\ \ \phi 3 - \tilde{z}1\ \ \phi 1 \qquad \phi 9 = \tilde{x}1\ \ \phi 2 - \tilde{y}1\ \ \phi 1$$

$$\tilde{x}1 = \frac{x1}{\sqrt{g}|\nabla k|} \qquad \tilde{y}1 = \frac{y1}{\sqrt{g}|\nabla k|} \qquad \tilde{z}1 = \frac{z1}{\sqrt{g}|\nabla k|}$$

$$\tilde{x}2 = \frac{x2}{\sqrt{g}|\nabla k|} \qquad \tilde{y}2 = \frac{y2}{\sqrt{g}|\nabla k|} \qquad \tilde{z}2 = \frac{z2}{\sqrt{g}|\nabla k|}$$

$$|\nabla k| = \sqrt{k_x^2 + k_y^2 + k_z^2} \qquad \text{and} \qquad \phi = \frac{4\tilde{c}}{\beta}(\beta + \tilde{\theta}_k(\tilde{\theta}_k - 2\tilde{k}_t)),$$

$(x1, y1, z1)$ and $(x2, y2, z2)$ are the diagonal vectors on the k = constant face [9]. In this section, a tilde over a quantity denotes that the metrics used in computing that quantity are normalized with the area of the cell face. The left and right eigenvectors of the flux Jacobians K are obtained as $T_k = MP_k$ and $T_k^{-1} = P_k^{-1}M^{-1}$ where the left eigenvectors are given by the rows of T_k^{-1}, and the right eigenvectors are given by the columns of T_k respectively. The matrices T_k and T_k^{-1} are

$$T_k = \begin{bmatrix} 0 & 0 & \tilde{x}^- & -\tilde{c}^+ \\ \tilde{x}1 & \tilde{x}2 & \tilde{k}_x + \dfrac{u'\tilde{\lambda}_k^3}{\beta} & \tilde{k}_x + \dfrac{u'\tilde{\lambda}_k^4}{\beta} \\ \tilde{y}1 & \tilde{y}2 & \tilde{k}_y + \dfrac{v'\tilde{\lambda}_k^3}{\beta} & \tilde{k}_y + \dfrac{v'\tilde{\lambda}_k^4}{\beta} \\ \tilde{z}1 & \tilde{z}2 & \tilde{k}_z + \dfrac{w'\tilde{\lambda}_k^3}{\beta} & \tilde{k}_z + \dfrac{w'\tilde{\lambda}_k^4}{\beta} \end{bmatrix}$$

$$T_k^{-1} = \frac{1}{\phi} \begin{bmatrix} \dfrac{2\tilde{c}}{\beta}(-u'\phi4 + v'\phi5 - w'\phi6) & 2\tilde{c}\phi4 & -2\tilde{c}\phi5 & 2\tilde{c}\phi6 \\ \dfrac{2\tilde{c}}{\beta}(u'\phi7 - v'\phi8 - w'\phi9) & -2\tilde{c}\phi7 & 2\tilde{c}\phi8 & -2\tilde{c}\phi9 \\ \dfrac{2}{\beta}(\beta + \tilde{\lambda}_k^4(\tilde{\theta}_k - 2\tilde{k}_t)) & 2\tilde{c}^+\tilde{k}_x & 2\tilde{c}^+\tilde{k}_y & 2\tilde{c}^+\tilde{k}_z \\ -\dfrac{2}{\beta}(\beta + \tilde{\lambda}_k^3(\tilde{\theta}_k - 2\tilde{k}_t)) & -2\tilde{c}^-\tilde{k}_x & -2\tilde{c}^-\tilde{k}_y & -2\tilde{c}^-\tilde{k}_z \end{bmatrix}$$

The quantity $T\Lambda^- T^{-1}\delta q$ which is required in the Roe flux formulation [10] is given by

$$T\Lambda^- T^{-1}\delta q = \begin{bmatrix} \lambda_k^4 r_{14} R_4 \\ \lambda_k^4 r_{24} R_4 - \lambda_k^1(r_{24}R_4 + r_{23}R_3 - \delta u) \\ \lambda_k^4 r_{34} R_4 - \lambda_k^1(r_{34}R_4 + r_{33}R_3 - \delta v) \\ \lambda_k^4 r_{44} R_4 - \lambda_k^1(r_{44}R_4 + r_{43}R_3 - \delta w) \end{bmatrix}$$

where

$$R_3 = l_{31}\delta p + l_{32}\delta u + l_{33}\delta v + l_{34}\delta w;$$
$$R_4 = l_{41}\delta p + l_{42}\delta u + l_{43}\delta v + l_{44}\delta w;$$

$(l_{31}, ..., l_{34})$ and $(l_{41}, ..., l_{44})$ are the 3rd and 4th left eigenvectors (that is, 3rd and 4th rows of T_k^{-1}); and $(r_{13}, ..., r_{43})^{\mathrm{T}}$ and $(r_{14}, ..., r_{44})^{\mathrm{T}}$ are the 3rd and 4th right eigenvectors (that is, 3rd and 4th columns of T_k). The quantity δq is given by $\delta q = q_R - q_L$ where q_R and q_L are defined using a MUSCL type approach [11, 14].

A requirement of the theory behind this numerical scheme is that the first order fluxes satisfy the property U defined by Roe [10]. It can be easily verified, by direct substitution, that the Roe averages, defined by $\overline{\phi} = (\phi_i + \phi_{i+1})/2$ where ϕ is any flow variable and the components of $\mathbf{w}$ at cell centers i and $i+1$ are taken to be the same as that at the cell face $i + 1/2$, satisfy the relation $F_{i+1} - F_i = A(\overline{\phi})[Q_{i+1} - Q_i]$ where F is the flux and A is the flux Jacobian.

A.2. *Relative-Velocity Procedure*

Equations (2.7) and (2.8) can now be cast into the numerical vector form which results in the same form as (A.1). However, the interpretation of the various symbols denoting the velocity components are as follows: u, v, w are the components of the relative velocity vector ($\mathbf{v}$) with respect to a Cartesian coordinate system and u', v', and w' are the Cartesian components of the vector $\mathbf{v} - 2\mathbf{w}$. (Note that $\mathbf{w}$ denotes the vector $-\mathbf{\Omega} \times \mathbf{r}$, whereas w denotes a Cartesian component of the absolute or relative velocity vector depending upon the context.) The analysis of the previous section carries through and the flux Jacobians as well as eigenvectors retain the same form as that given by the matrices K, κ, T_k, and T_k^{-1} respectively. The eigenvalues of this system are given by

$$\left.\begin{aligned} \lambda_{1,2} &= \theta_k \\ \lambda_3 &= \theta_k - k_t + c \\ \lambda_4 &= \theta_k - k_t - c \end{aligned}\right\} \tag{A.5}$$

where

$$\theta_k = uk_x + vk_y + wk_z \qquad \text{and}$$
$$c = \sqrt{(\theta_k - k_t)^2 + \beta(k_x^2 + k_y^2 + k_z^2)}.$$

Note that the definition of θ_k is different from that of the absolute-velocity procedure. k_t is the same as before.

It follows from the above analysis that with a minimum code modification of a few lines, one can solve for either the absolute velocity components or the relative velocity components.

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